ON THE PLACING OF A GYROSCOPIC COMPASS INTO A MERIDIAN DURING THE ACCELERATION OF THE ROTORS OF THE GYROSCOPES

(O PRIVEDENII GIROSKOPICHESKOGO KOMPASA V MERIDIAN VO VREMIA RAZGONA ROTOROV GIROSKOPOV)

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The placement of a gyrocompass into a meridian after it has been started is usually accomplished by means of application to the gyroscope of additional external forces. The selection of the law of control of these forces may be subjected to the requirement of placing the gyrocompass into a meridian within an interval of time given in advance [1]. To accelerate the readiness of the gyrocompass it is advisable to start placing it into the meridian when the rotors of the gyroscopes are still being accelerated. For a gyrocompass with variable kinetic (strictly speaking, proper) moments of gyroscopes, we are led to the problem analogous to the one considered in paper [1], mentioned above.

1. The equations of motion of a gyrocompass with variable proper moments of gyroscopes are of the following form:

$$A\ddot{\alpha} + \dot{H}\beta + H\dot{\beta} + HU\cos\varphi\,\alpha = 0$$

$$B\ddot{\beta} - H\dot{\alpha} + lP\beta + lP(1-\rho)\vartheta = HU\sin\varphi + Q(t)$$

$$\dot{\vartheta} + F\beta + F\vartheta = 0 \qquad (1.1)$$

Here α is the angle of rotation of the gyroscope in the azimuth, β is the angle of elevation of the north diameter of the gyrosphere above the horizontal plane, ϑ is the angle of inclination of the liquid level in the hydraulic damper above the equator plane of the gyrosphere, H is the resultant proper moment of the gyroscopes mounted in the gyrosphere (the law of its variation with time $H = H(t)$ is assumed to be known), lP is the static moment of the gyrosphere, A and B are the moments of inertia of the gyrosphere with respect to correspondent axes, U is the angular velocity of the daily rotation of the earth, ϕ is the latitude of the point of observation, $Q(t)$ is the supplementary generalized external force which represents a moment with respect to the eastern diameter of the gyrosphere, applied for accelerated placing of the gyrocompass into a meridian (the law of variation of this external force with respect to time is to be determined).

Limiting ourselves to a study of precessional motion of the gyrocompass, we omit in Equations (1.1) the inertia terms $A\ddot{\alpha}$ and $B\ddot{\beta}$. Equations (1.1) are to be reduced to the form

$$\begin{gathered}\dot{\alpha}-\frac{lP}{H}\beta-\frac{lP}{H}(1-\rho)\vartheta=-U\sin\varphi-\frac{Q(t)}{H}\\ \dot{\beta}+U\cos\varphi\,\alpha+\frac{\dot{H}}{H}\beta=0,\qquad \dot{\vartheta}+F\beta+F\vartheta=0\end{gathered}\tag{1.2}$$

Introducing the matrices

$$\begin{gathered}z=\left\|\begin{matrix}z_1\\z_2\\z_3\end{matrix}\right\|=\left\|\begin{matrix}\alpha\\\beta\\\vartheta\end{matrix}\right\|,\qquad s(t)=\|s_i(t)\|=\left\|\begin{matrix}-U\sin\varphi-Q(t)/H\\0\\0\end{matrix}\right\|\\ a(t)=\|a_{jk}(t)\|=\left\|\begin{matrix}0 & -lP/H & (-lP/H)(1-\rho)\\ U\cos\varphi & \dot{H}/H & 0\\ 0 & F & F\end{matrix}\right\|\end{gathered}\tag{1.3}$$

we replace the system of scalar equations (1.2) by equivalent matrix equation

$$\dot{z}+a(t)z=s(t)\tag{1.4}$$

The general solution of Equation (1.4) is of the form:

$$z(t)=N(t,0)z(0)+\int_0^t N(t,\tau)s(\tau)\,d\tau\tag{1.5}$$

where

$$N(t,\tau)=\theta(t)\theta^{-1}(\tau)\tag{1.6}$$

and $\theta(t)$ is the fundamental matrix for the homogeneous matrix equation, obtained from (1.4) with $s(t)\equiv 0$.

Since $s_2(t)\equiv s_3(t)\equiv 0$, the elements of the matrix z will be

$$z_j(t)=\sum_{k=1}^{3}N_{jk}(t,0)z_k(0)+\int_0^t N_{j1}(t,\tau)s_1(\tau)\,d\tau\qquad (j=1,2,3)\tag{1.7}$$

Expressions (1.7) determine the motion of a gyroscopic compass for arbitrary initial conditions $z_k(0)$.

2. We proceed to the selection of a law of variation of the supple-

mentary external force $Q(t)$, with respect to time, using the requirement that at the instant of time $t = T$ the gyrocompass should be placed in a meridian [2]. We shall assume thereby that at the instant of time $t = T$ the acceleration of rotors of the gyroscopes is already terminated. The requirement advanced above is contained in the fact that at the instant of time $t = T$ the generalized coordinates of the gyroscope have the following values:

$$z_1(T) = z_1^*, \qquad z_2(T) = z_2^*, \qquad z_3(T) = z_3^* \tag{2.1}$$

where

$$z_1^* = 0, \qquad z_2^* = \frac{H_n U \sin\varphi}{\rho l P}, \qquad z_3^* = -\frac{H_n U \sin\varphi}{\rho l P} \tag{2.2}$$

The quantities (2.2) determine an equilibrium state (strictly speaking a stationary motion) of the gyroscope with a steady value of angular velocity of spin of the gyroscopes which corresponds to a stationary value of the H_n resultant proper moment of the gyroscopes.

Conditions (2.1) are equivalent to the following relations, which may be obtained with the aid of (1.7):

$$\sum_{k=1}^{3} N_{jk}(T, 0)\, z_k(0) + \int_0^T N_{j1}(T, \tau)\, s_1(\tau)\, d\tau = z_j^* \qquad (j = 1, 2, 3) \tag{2.3}$$

where, according to (1.3),

$$s_1(\tau) = -U\sin\varphi - \frac{Q(\tau)}{H(\tau)}$$

Introducing the notation

$$R_j(T) = \sum_{k=1}^{3} N_{jk}(T, 0)\, z_k(0) - U\sin\varphi \int_0^T N_{j1}(T, \tau)\, d\tau - z_j^* \qquad (j = 1, 2, 3) \tag{2.4}$$

we reduce relations (2.3) to the following form:

$$\int_0^T N_{j1}(T, \tau) \frac{Q(\tau)}{H(\tau)}\, d\tau = R_j(T) \qquad (j = 1, 2, 3) \tag{2.5}$$

The interval of time $(0, T)$ is now subdivided into three equal intervals $(0, t_1)$, (t_1, t_2), (t_2, T) and the function $Q(t)$ is assumed to be a stepwise one, conserving its values along these intervals of time. Let us designate these values by $Q(0)$, $Q(t_1)$ and $Q(t_2)$, respectively.

Relations (2.5) may now be represented in the form

$$c_j^{(0)}Q(0)+c_j^{(1)}Q(t_1)+c_j^{(2)}Q(t_2)=R_j(T) \qquad (j=1,\ 2,\ 3) \tag{2.6}$$

where

$$c_j^{(0)}=\int_0^{t_1}\frac{N_{j1}(T,\ \tau)}{H(\tau)}d\tau, \qquad c_j^{(1)}=\int_{t_1}^{t_2}\frac{N_{j1}(T,\ \tau)}{H(\tau)}d\tau, \quad c_j^{(2)}=\int_{t_2}^{T}\frac{N_{j1}(T,\ \tau)}{H(\tau)}d\tau \tag{2.7}$$

From Equations (2.6) we obtain

$$Q(0)=\frac{\Delta_1}{\Delta}, \qquad Q(t_1)=\frac{\Delta_2}{\Delta}, \qquad Q(t_2)=\frac{\Delta_3}{\Delta} \tag{2.8}$$

where

$$\Delta_1=\begin{vmatrix} R_1(T) & c_1^{(1)} & c_1^{(2)} \\ R_2(T) & c_2^{(1)} & c_2^{(2)} \\ R_3(T) & c_3^{(1)} & c_3^{(2)} \end{vmatrix}, \qquad \Delta_2=\begin{vmatrix} c_1^{(0)} & R_1(T) & c_1^{(2)} \\ c_2^{(0)} & R_2(T) & c_2^{(2)} \\ c_3^{(0)} & R_3(T) & c_3^{(2)} \end{vmatrix}$$

$$\Delta_3=\begin{vmatrix} c_1^{(0)} & c_1^{(1)} & R_1(T) \\ c_2^{(0)} & c_2^{(1)} & R_2(T) \\ c_3^{(0)} & c_3^{(1)} & R_3(T) \end{vmatrix}, \qquad \Delta=\begin{vmatrix} c_1^{(0)} & c_1^{(1)} & c_1^{(2)} \\ c_2^{(0)} & c_2^{(1)} & c_2^{(2)} \\ c_3^{(0)} & c_3^{(1)} & c_3^{(2)} \end{vmatrix} \tag{2.9}$$

Expressions (2.8) determine the law in accordance to which $Q(t)$ must vary such that at the instant of time $t = T$ the gyrocompass be in a meridian.

In calculating the values (2.8), the functions $N_{j1}(T,\ \tau)$ $(j = 1,\ 2,\ 3)$. the expressions((2.4) and (2.7) are assumed to be known on the interval $0 \leqslant \tau \leqslant T$; they represent the elements of the matrix function of weight $N(T,\ \tau)$ for a fixed value $t = T$. The values $N_{j\xi}(T,\ 0)$ $(j,\ \xi = 1,\ 2,\ 3)$, entering into expressions (2.4), are also assumed to be known; they represent the values of the elements of the matrix function of weight $N(t,\ \tau)$ for $t = T$, $\tau = 0$.

As is known

$$N_{j\xi}(T,\ \tau)=Z_\xi(\tau) \tag{2.10}$$

where $Z_\xi(\tau)$ are the integrals of the system of equations (1.2) constructed for the conjugate system of equations

$$\frac{dZ_\xi}{d\tau}-\sum_{k=1}^{3}a_{k\xi}(\tau)Z_k=0 \qquad (\xi=1,\ 2,\ 3) \tag{2.11}$$

which for $\tau = T$ take on the values

$$Z_j(T) = 1, \qquad Z_k(T) = 0 \qquad (k \neq j) \tag{2.12}$$

Equations (2.11), after substitution of the values of the coefficients $a_{k\xi}(\tau)$ in accordance with (1.3), take on the form:

$$\begin{aligned} &\frac{dZ_1}{d\tau} - U\cos\varphi\, Z_2 = 0 \\ &\frac{dZ_2}{d\tau} + \frac{lP}{H(\tau)} Z_1 - \frac{\dot{H}(\tau)}{H(\tau)} Z_2 - FZ_3 = 0 \\ &\frac{dZ_3}{d\tau} + \frac{lP(1-\rho)}{H(\tau)} Z_1 - FZ_3 = 0 \end{aligned} \tag{2.13}$$

Let us now introduce a new independent variable σ with the aid of the relation

$$\sigma = T - \tau \tag{2.14}$$

and let us use the notation

$$\begin{aligned} &Z_\xi(\tau) = Z_\xi(T-\sigma) = Y_\xi(\sigma) \\ &H(\tau) = H(T-\sigma) = h(\sigma) \\ &\frac{H(\tau)}{H(\tau)} = M(\tau) = M(T-\sigma) = m(\sigma) \end{aligned} \tag{2.15}$$

Taking into account that

$$\frac{dZ_\xi}{d\tau} = \frac{dY_\xi}{d\sigma}\frac{d\sigma}{d\tau} = -\frac{dY_\xi}{d\sigma} \tag{2.16}$$

we may transform the system of equations (2.13) to the following form:

$$\begin{aligned} &\frac{dY_1}{d\sigma} + U\cos\varphi\, Y_2 = 0 \\ &\frac{dY_2}{d\sigma} - \frac{lP}{h(\sigma)} Y_1 + m(\sigma) Y_2 + FY_3 = 0 \\ &\frac{dY_3}{d\sigma} - \frac{lP(1-\rho)}{h(\sigma)} Y_1 + FY_3 = 0 \end{aligned} \tag{2.17}$$

Conditions (2.12) in accordance with (2.15) take on the form:

$$Y_j(0) = 1, \qquad Y_k(0) = 0 \qquad (k \neq j) \tag{2.18}$$

Thus, in order to determine the functions $N_{j\xi}(T, \tau)$ it is necessary to integrate the system of equations (2.17) with initial conditions (2.18). The integration should be carried out for three systems of initial conditions which can be obtained from (2.18), in accordance with the values $j = 1, 2, 3$.

t sec	y_1	$10^3 y_2$	$10^3 y_3$	t sec	y_1	$10^3 y_2$	$10^3 y_3$
0	0.3000	4.000	4.000	950	−0.0546	1.218	1.146
50	0.2329	0.936	3.552	1000	−0.0254	1.285	0.973
100	0.1739	−0.392	3.282	1050	0.0036	1.299	0.809
150	0.1208	−1.121	3.102	1100	0.0323	1.262	0.657
200	0.0717	−1.553	2.976	1150	0.0608	1.174	0.521
250	0.0254	−1.799	2.883	1200	0.0888	1.036	0.404
300	−0.0188	−1.911	2.810	1250	0.0829	0.877	0.305
350	−0.0614	−1.914	2.746	1300	0.0767	0.730	0.225
400	−0.1026	−1.823	2.683	1350	0.0700	0.596	0.161
450	−0.1426	−1.647	2.615	1400	0.0631	0.473	0.111
500	−0.1815	−1.399	2.536	1450	0.0558	0.364	0.073
550	−0.2193	−1.060	2.441	1500	0.0483	0.269	0.045
600	−0.2561	−0.657	2.327	1550	0.0405	0.188	0.025
650	−0.2282	−0.242	2.191	1600	0.0326	0.121	0.013
700	−0.1998	0.125	2.037	1650	0.0246	0.068	0.005
750	−0.1711	0.444	1.868	1700	0.0164	0.031	0.001
800	−0.1422	0.713	1.691	1750	0.0082	0.008	0.0001
850	−0.1130	0.932	1.509	1800	0.0000	0.000	0.000
900	−0.0838	1.101	1.326				

3. As an example let us consider a gyrocompass with the following parameters:

$$lP = 6400 \text{ gr cm} \quad H_n = 155000 \text{ gr cm sec} \quad \rho = 0.4, \quad F = 1.5\cdot 10^{-3} \text{ sec}^{-1}$$

The variation of the proper moment of the gyroscopes with respect to time takes place in accordance with the law

$$H(t) = H_n(1 - \eta e^{-\gamma t}) \qquad (\eta = 0.7, \ \gamma = 0.005)$$

If the latitude ϕ of the place of observation is equal to 60°, then

$$U\cos\varphi = 3.646\cdot 10^{-5} \text{ sec}^{-1}$$

The interval of time during which the gyrocompass should be placed into the meridian is T = 1800 sec. The initial deviations are

$$z_1(0) = 0.3, \qquad z_2(0) = 7.82\cdot 10^{-3}, \qquad z_3(0) = 0.18\cdot 10^{-3}$$

With this data the values $Q(0)$, $Q(t_1)$, $Q(t_2)$ are the following:

$$Q(0) = 113.47, \qquad Q(t_1) = -77.78, \qquad Q(t_2) = 25.54 \ [\Gamma\text{cm}]$$

The process of placing the gyrocompass into the meridian is given by the table of the values of functions

$$y_j(t) = z_j(t) - z_j^* \qquad (j = 1, 2, 3) \tag{2.19}$$

In accordance with the values of the parameters from (2.2) we have here

$$z_1^* = 0, \qquad z_2^* = 3.82\cdot 10^{-3}, \qquad z_3^* = -3.82\cdot 10^{-3}$$

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